APPROXIMATING L^2 INVARIANTS OF AMENABLE COVERING SPACES: A HEAT KERNEL APPROACH.

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Abstract. In this paper, we prove that the L^2 Betti numbers of an amenable covering space can be approximated by the average Betti numbers of a regular exhaustion, under some hypotheses. We also prove that some L^2 spectral invariants can be approximated by the corresponding average spectral invariants of a regular exhaustion. The main tool which is used is a generalisation of the "principle of not feeling the boundary" (due to M. Kac), for heat kernels associated to boundary value problems.

Introduction

There has been important work done recently on approximating L^2 invariants on *residually finite* covering spaces due to Lück [Lu] (see eg. [Don1] for earlier results). This paper is the first of a couple of papers, where we study the approximation of L^2 invariants on *amenable* covering spaces, using heat kernel techniques. In a sequel paper [DM], we continue this study, but now using combinatorial techniques instead and where we manage to prove the main conjecture stated in this paper.

Let (M, g) be a compact Riemannian manifold and $\Gamma \to \widehat{M} \to M$ be a Galois covering space of M. Recall that *Cheeger's isoperimetric constant* is defined as

$$h(\widehat{M}, g) = \inf \left\{ \frac{\mathrm{vol}_{n-1}(\partial D)}{\mathrm{vol}_n(D)} : D \quad \text{is open in} \quad \widehat{M} \quad \text{and} \quad \partial D \quad \text{is smooth} \right\}.$$

Now let g' be another Riemannian metric on M. Since g and g' are quasi-isometric, one sees that there is a positive constant C such that

$$\frac{1}{C} h(\widehat{M}, g') \le h(\widehat{M}, g) \le C h(\widehat{M}, g').$$

A Galois covering space $\Gamma \to \widehat{M} \to M$ is said to be *amenable* if $h(\widehat{M}, g) = 0$. By the previous discussion, this is independent of the choice of the Riemannian metric g on M. In fact, it turns out that $\widehat{M}$ is amenable if and only if Γ is amenable. There are many examples of amenable groups viz.

(1) Abelian groups;
(2) nilpotent groups and solvable groups;
(3) groups of subexponential growth;
(4) subgroups, quotient groups and extensions of amenable groups;

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(5) direct limits of amenable groups.

A sequence of open, connected, relatively compact subsets $D_k, k = 1, 2, \dots$ will be called an *exhaustion* of $\widehat{M}$ if

(1) $\overline{D}_k \subset D_{k+1} \quad \forall k \geq 1$
(2) $\bigcup_{k \geq 1} D_k = \widehat{M}$.

If the group Γ is amenable, we can choose an exhaustion such that $\lim_{k\to\infty} \frac{\mathrm{vol}_{n-1}(\partial D_k)}{\mathrm{vol}_n(D_k)} = 0$ (cf. [Ad]). This is essentially a combinatorial fact. By choosing an exhaustion of the Cayley graph of Γ and a corresponding exhaustion of $\widehat{M}$ by unions of translates of a fundamental domain with piecewise smooth boundary, and smoothing the boundaries in a uniform way [AdSu, Proposition 3.2], [CG2], we can assume that our exhaustion has the following properties.

(1) For every fixed $\delta > 0$,

$$\lim_{k\to\infty} \frac{\mathrm{vol}_n(\partial_\delta D_k)}{\mathrm{vol}_n(D_k)} = 0, \tag{0.1}$$

where $\partial_\delta D_k = \{x \in \widehat{M} : d(x, \partial D_k) < \delta\}$.

(2) The second fundamental forms of hypersurfaces ∂D_k are uniformly bounded by a constant independent of k.
(3) The boundaries ∂D_k are uniformly collared in the following sense. There exists a constant ρ such that the mapping

$$\Phi_k : \partial D_k \times [0, \rho] \longrightarrow D_k$$
$$\Phi(x, t\xi) = \exp_x(t\xi)$$

is a diffeomorphism onto its image. Here $x \in \partial D_k$, $t \in [0, \rho]$ and ξ is the unit interior normal vector field on ∂D_k.

We shall call an exhaustion $\{D_k\}_{k=1}^{\infty}$ satisfying these three properties a *regular exhaustion* of $\widehat{M}$.

Let $b^j(D_k, \partial D_k)$ denote the j^{th} Betti number of D_k relative to the boundary and $b^j(D_k)$ denote the j^{th} Betti number of D_k. Let $b^j_{(2)}(\widehat{M}, \Gamma)$ denote the j^{th} L^2 Betti number of $\widehat{M}$, then one of our main results is

Theorem 0.1. *Let $\Gamma \to \widehat{M} \to M$ be a noncompact Galois cover of a compact manifold M which is amenable. Let $\{D_k\}_{k=1}^{\infty}$ be a regular exhaustion of $\widehat{M}$, then for $j \geq 0$*

a)

$$\limsup_{k\to\infty} \frac{vol(M)}{vol(D_k)} b^j(D_k, \partial D_k) \leq b^j_{(2)}(\widehat{M}, \Gamma).$$

and

$$\limsup_{k\to\infty} \frac{vol(M)}{vol(D_k)} b^j(D_k) \leq b^j_{(2)}(\widehat{M}, \Gamma).$$

In particular, equality holds whenever $b^j_{(2)}(\widehat{M}, \Gamma) = 0$.

b) *More generally, one has for* $N \geq 0$

$$\limsup_{k\to\infty} \sum_{j=0}^{N} (-1)^{N-j} \frac{vol(M)}{vol(D_k)} b^j(D_k, \partial D_k) \leq \sum_{j=0}^{N} (-1)^{N-j} b^j_{(2)}(\widehat{M}, \Gamma).$$

and

$$\limsup_{k\to\infty} \sum_{j=0}^{N} (-1)^{N-j} \frac{vol(M)}{vol(D_k)} b^j(D_k) \leq \sum_{j=0}^{N} (-1)^{N-j} b^j_{(2)}(\widehat{M}, \Gamma).$$

Moreover, there is equality when $N = n = \dim M$.

We remark that $b^j_{(2)}(\widehat{M}, \Gamma) = 0$, trivially when $j = 0, n$, but in many instances for other values of j as well.

We conjecture that under the hypothesis of the theorem, one always has equality.

Conjecture. *Let* $\Gamma \to \widehat{M} \to M$ *be a noncompact Galois cover of a compact manifold* M *which is amenable. Let* $\{D_k\}_{k=1}^{\infty}$ *be a regular exhaustion of* $\widehat{M}$*, then for* $j \geq 0$

$$\lim_{k\to\infty} \frac{vol(M)}{vol(D_k)} b^j(D_k, \partial D_k) = \lim_{k\to\infty} \frac{vol(M)}{vol(D_k)} b^j(D_k) = b^j_{(2)}(\widehat{M}, \Gamma).$$

In other words, the conjecture says that on amenable covering spaces, the averaged absolute cohomology and the averaged relative cohomology yield the same limit. The next theorem establishes this conjecture for manifolds of dimension less than or equal to four, under a mild hypothesis.

Theorem 0.2. *Let* $\Gamma \to \widehat{M} \to M$ *be a noncompact Galois cover of a compact manifold* M *which is amenable and oriented.*

a) *Suppose that* $dimM = 2$*. Then the conjecture is true.*

b) *Suppose that* $dimM = 3$ *or* 4 *and that* $dimH^1(\widehat{M}) < \infty$*. Then the conjecture is true.*

Remarks. There are many amenable Galois covering spaces of a surface of genus ≥ 1. One sees this as follows. Since Galois covering spaces of a surface Σ_g of of genus g correspond to normal quotient groups of the fundamental group $\pi_1(\Sigma_g) = \Gamma_g$, the question of determining amenable Galois covering spaces of Σ_g becomes a purely group theoretic question. Some examples are as follows; one can consider the universal homology covering $\mathbb{Z}^{2g} \to \widehat{\Sigma_g} \to \Sigma_g$ and its quotients. A method of producing non-Abelian amenable covering spaces is to first consider the natural homomorphism of the fundamental group Γ_g onto the free group on g generators F_g, and then one knows that there are many non-Abelian amenable groups on two or more generators, i.e. quotients of F_g. For example $\mathbb{Z}/2 * \mathbb{Z}/2$, the integer Heisenberg groups etc. By the previous discussion, all these correspond to non-Abelian amenable covering spaces of surfaces. We remark that it follows from Atiyah's L^2-index theorem [At] that $b^1_{(2)} \neq 0$ for all these spaces.

We also establish the conjecture under the hypothesis that the heat kernel converges uniformly to the harmonic projection. More precisely, one has the following.

Theorem 0.3. *Let $\Gamma \to \widehat{M} \to M$ be a noncompact Galois cover of a compact manifold M which is amenable.*

a) *Suppose that there is a function $f : [1,\infty] \to [0,\infty)$ which converges to zero as $t \to \infty$, such that for $j \geq 0$*

$$\frac{vol(M)}{vol(D_k)}\Big[Tr(e^{-t\Delta_j^{(k)}}) - b^j(D_k, \partial D_k)\Big] \leq f_j(t)$$

for all $k \in \mathbb{N}$ and for all $t \geq 1$. Here $\Delta_j^{(k)}$ denotes the Laplacian on D_k with the relative boundary conditions. Then the conjecture is true.

b) *Suppose that there is a function $f : [1,\infty] \to [0,\infty)$ which converges to zero as $t \to \infty$, such that for $j \geq 0$*

$$\frac{vol(M)}{vol(D_k)}\Big[Tr(e^{-t\Delta_j^{(k)}}) - b^j(D_k)\Big] \leq f_j(t)$$

for all $k \in \mathbb{N}$ and for all $t \geq 1$. Here $\Delta_j^{(k)}$ denotes the Laplacian on D_k with the absolute boundary conditions. Then the conjecture is true.

In the first section, we recall some preliminary material on von Neumann algebras and L^2 cohomology. In the second section, we prove the main heat kernel estimates, which are quantitative versions of the principle of not feeling the boundary. In section 3, we give a heat kernel proof of Theorem 0.1. We also give the proofs of Theorems 0.2 and 0.3 in this section. Theorem 0.2 uses Theorem 0.1 together with an index theorem. We also present non-examples to the analogue of Theorem 0.2 for non-amenable covering spaces. A key result in the paper of Adachi and Sunada [AdSu] is an approximation property for the spectral density functions, with Dirichlet boundary conditions. They also conjecture in section 1 of their paper that such an approximation property should also hold for the spectral density functions, with Neumann boundary conditions. Proposition 3.1 proves their conjecture, and much more, as complete results are proved for differential forms, and not just for functions. In section 4, we derive some useful corollaries of the Theorems which were proved in the earlier sections. We also prove miscellaneous results on the spectrum of the Laplacian on amenable covering spaces.

1. The von Neumann trace and L^2 cohomology

In this section,we briefly review results on the von Neumann trace and L^2 cohomology which are used in the paper. We shall assume that $\Gamma \to \widehat{M} \to M$ be a noncompact Galois cover of a compact Riemannian manifold (M, g). Then the Riemannian metric on M lifts to a Riemannian metric on the Galois cover $\widehat{M}$, and one can define the spaces of L^2 differential forms on $\widehat{M}$, $\Omega^\bullet_{(2)}(\widehat{M})$. Let d denote the deRham exterior derivative. One can define the space of *closed* L^2 differential p-forms

$$Z^p(\widehat{M}) = \{\eta \in \Omega^p_{(2)}(\widehat{M}) : d\eta = 0\}.$$

It is not hard to see that $Z^p(\widehat{M})$ is a closed subspace of $\Omega^p_{(2)}(\widehat{M})$. One can also define the space of *exact* L^2 differential p-forms

$$B^p(\widehat{M}) = \{\eta \in \Omega^p_{(2)}(\widehat{M}) : \eta = d\phi \ \text{ for some } \ \phi \in \Omega^{p-1}_{(2)}(\widehat{M})\}.$$

In the example when M is the circle, one can see that $B^p(\widehat{M})$ is *not* a closed subspace of $\Omega^p_{(2)}(\widehat{M})$. It is natural then to take the closure, $\overline{B^p(\widehat{M})}$, and to define the *reduced* L^2 *cohomology* of $\widehat{M}$ as

$$\bar{H}^p(\widehat{M}) = Z^p(\widehat{M})/\overline{B^p(\widehat{M})}.$$

It is clear that $\bar{H}^p(\widehat{M})$ is Γ-invariant. Dodziuk [Dod1] proved that they depend only on the homotopy type of M.

Let δ denote the L^2 adjoint of d. Then the Laplacian Δ_p acting on $\Omega^p_{(2)}(\widehat{M})$ is

$$\Delta_p = d\delta + \delta d.$$

One can define the space of L^2 *harmonic forms* as

$$\mathcal{H}^p(\widehat{M}) = \{\eta \in \Omega^p_{(2)}(\widehat{M}) : \Delta_p \eta = 0\}.$$

It is clear that $\mathcal{H}^p(\widehat{M})$ is Γ-invariant and one can show that $\mathcal{H}^p(\widehat{M})$ and $\bar{H}^p(\widehat{M})$ are Γ-isomorphic Hilbert spaces. One also has the Kodaira-Hodge decomposition (see [GS])

$$\Omega^p_{(2)}(\widehat{M}) = \mathcal{H}^p(\widehat{M}) \oplus \overline{d\Omega^{p-1}_c(\widehat{M})} \oplus \overline{\delta\Omega^{p+1}_c(\widehat{M})}$$

where $\Omega^\bullet_c(\widehat{M})$ denotes the space of compactly supported differential forms on $\widehat{M}$.

The commutant of Γ action on $\Omega^p_{(2)}(\widehat{M})$ is a von Neumann algebra with trace given by

$$\mathrm{Tr}_\Gamma(L) = \int_M \mathrm{tr}(L(x,x))d\mu(x),$$

where L is an operator in the commutant of Γ with smooth integral kernel $L(x,y)$, $x, y \in \widehat{M}$. Therefore, there is a well defined dimension function $\dim_\Gamma$, known as the *von Neumann dimension* function, which is defined on Γ-invariant closed subspaces of $\Omega^p_{(2)}(\widehat{M})$ and which takes values in $\mathbb{R}$. In particular, consider the spectral projections $E_p(\lambda) = \chi_{[0,\lambda]}(\Delta_p)$. By elliptic regularity theory, the integral kernel of $E_p(\lambda)$ is smooth, and therefore one can define the *von Neumann spectral function* for Δ_p as

$$N_{p,\Gamma}(\lambda) := \dim_\Gamma(\mathrm{Im}E_p(\lambda)) = \mathrm{Tr}_\Gamma(E_p(\lambda)) < \infty.$$

In particular, the L^2 Betti numbers are defined as the von Neumann dimensions of the orthogonal projection onto the L^2 harmonic forms on $\widehat{M}$, $E_p(0)$ (cf. [At], [Dod1]), i.e.

$$b^p_{(2)}(\widehat{M},\Gamma) = \dim_\Gamma(\mathcal{H}^p(\widehat{M})) = \dim_\Gamma(\bar{H}^p(\widehat{M})) = N_{p,\Gamma}(0).$$

They have the following key properties.

a) *Homotopy invariance property*, [Dod1]. For all $p \geq 0$, $b^p_{(2)}(\widehat{M},\Gamma)$ depends only on the homotopy type of the closed manifold M.
b) *Euler characteristic property*, [At].

$$\sum_{p=0}^n (-1)^p b^p_{(2)}(\widehat{M},\Gamma) = \chi(M).$$

c) *Finite covers.* If $\widehat{M}$ is a finite cover of M, then for all $p \geq 0$

$$b^p_{(2)}(\widehat{M}, \Gamma) = \frac{b^j(\widehat{M})}{\#\Gamma}.$$

d) *Residually finite approximation*, [Lu]. Let $\ldots \Gamma_{m+1} \subset \Gamma_m \ldots \Gamma$ be a nested sequence of normal subgroups of Γ of finite index such that $\cap_{m\geq 1}\Gamma_m = \{e\}$. Then Γ is said to be a *residually finite* group and one has for all $p \geq 0$

$$b^p_{(2)}(\widehat{M}, \Gamma) = \lim_{m\to\infty} \frac{b^p(M_m)}{\#(\Gamma/\Gamma_m)}$$

where $\Gamma/\Gamma_m \to M_m \to M$ is the corresponding sequence of finite covers that "approximate" $\widehat{M}$.

2. The principle of not feeling the boundary

In this section, we shall assume that $\widehat{M} \to M$ is a noncompact Riemannian covering of a compact manifold M and $\{D_k\}_{k=1}^{\infty}$ is an exhaustion of $\widehat{M}$ satisfying conditions (2) and (3) in the definition of regular exhaustion. We emphasize that we do not require that the covering be amenable, i.e. that (0.1) be satisfied. We shall call the image $\Phi(\partial D_k \times [0, \rho])$ the collar of the boundary and will denote it by $B(\partial D_k)$. The key technical part of our paper is to prove a quantitative version of the following

Principle of not feeling the boundary. *Let $k_j(t, x, y)$ denote the heat kernel on L^2 j-forms on $\widehat{M}$ and $p^k_j(t, x, y)$ denote the heat kernel on D_k which is associated to either the relative boundary conditions or the absolute boundary conditions, then as $t \to 0$*

$$k_j(t, x, y) \sim p^k_j(t, x, y).$$

This is a generalisation of the well known principle on functions due to M. Kac [K]. The result below is weaker than what could be obtained by the method of [RS, Section 7]. The classical technique employed there is rather intricate and makes it difficult to identify the dependence of constants in the estimates on geometry. For applications that we have in mind, it is essential to have estimates valid uniformly for all sets D_k and the proof below does that.

Theorem 2.1. (Main heat kernel estimate) *Let $k_j(t, x, y)$ denote the heat kernel on L^2 j-forms on $\widehat{M}$ and $p^k_j(t, x, y)$ denote the heat kernel on D_k which is associated to either the relative boundary conditions or the absolute boundary conditions. There are positive constants C_1, C_2 depending only on the geometry of $\widehat{M}$, the second fundamental form of ∂D_k, on the choice of ρ and on T, such that for $x, y \in D_k \setminus B(\partial D_k)$, $t \in (0, T]$,*

$$|k_j(t, x, y) - p^k_j(t, x, y)| \leq C_1 e^{-C_2 \frac{D^2(x)+D^2(y)}{t}},$$

where $D_k(x) = d(x, \partial D_k)$.

To prove this theorem we study solutions of the heat equation in D_k which vanish identically for $t = 0$. The difference of heat kernels $\omega(x, t) = k_j(t, x, y) - p^k_j(t, x, y)$ for

a fixed y is such a solution as a consequence of Minakshisundaram-Pleijel asymptotic expansion of the heat kernels [RS]. The Weitzenböck identity,

$$\Delta\omega = \nabla^*\nabla\omega + F_p\omega,$$

where ∇ is the operator of covariant differentiation, ∇^* is its formal adjoint, and F_p is an algebraic operator depending only on the curvature of M, implies that

$$\left(\Delta + \frac{\partial}{\partial t}\right) e^{-\alpha t}|\omega|^2 = e^{-\alpha t}(-2\langle\nabla\omega, \nabla\omega\rangle + 2\langle\omega, F_p\omega\rangle - \alpha|\omega|^2).$$

Therefore, cf. [Dod3, Section 5], for $\alpha = \sup_{x\in M}|F_p|$,

$$u(x,t) = e^{-\alpha t}|\omega(x,t)|^2$$

satisfies

$$\left(\Delta + \frac{\partial}{\partial t}\right) u \leq 0 \tag{2.1}$$

i.e. $u(x,t)$ is a subsolution of the heat equation on functions. In particular, the weak maximum principle applies to u. This is used in the proof of the following lemma.

Lemma 2.2. *Let $u(x,t)$ be a positive subsolution of the heat equation on functions in D_k such that $u(x,0) \equiv 0$. Suppose*

$$N = \sup_{(z,t)\in\partial D_k\times[0,T]} u(z,t).$$

There exists constants $C_3, C_4 > 0$ independent of k such that if $D(x) \geq \rho$ then

$$u(x,t) \leq C_3 N e^{-C_4 D^2(x)/t}$$

for $t \in [0,T]$. The constants depend on T and on the local geometry but not on k.

Proof. Choose a smooth nonnegative function $\mu(r)$ on $[0,1]$ such that $\mu|[0,1/3] \equiv 1$ and $\mu|[2/3,1] \equiv 0$. For x with $D(x) \geq \rho$ define a cut-off function $h(y)$ as follows.

$$h(y) = \begin{cases} 0 & \text{if } d(x,y) \geq D(x), \\ 1 & \text{if } d(x,y) \leq D(x) - (2/3)\rho, \\ \mu(\frac{d(x,y)-D(x)+\rho}{\rho}) & \text{if } d(x,y) \in [D(x)-\rho, D(x)]. \end{cases}$$

Thus $h(y)$ vanishes near ∂D_k, and is identically equal to one near x. If $q(t,x,y) = k^0(t,x,y)$ is the heat kernel for functions on $\widehat{M}$, using the fact that u is a subsolution and

that q is a solution of the heat equation, we see that

$$\begin{aligned}
u(x,t) &= \int_0^t \frac{d}{d\tau}\int_{D_k} h(y)u(y,\tau)q(t-\tau,y,x)\,d\mu(y)\,d\tau \\
&= \int_0^t\int_{D_k}\left(h(y)\frac{du}{d\tau}(y,\tau)q(t-\tau,y,x) + h(y)u(y,\tau)\frac{dq}{d\tau}(t-\tau,y,x)\right)\,d\mu(y)\,d\tau \\
&\leq \int_0^t\int_{D_k}\left(-h(y)\Delta u(y,\tau)q(t-\tau,y,x) - h(y)u(y,\tau)\Delta_y q(t-\tau,y,x)\right)\,d\mu(y)\,d\tau \\
&= -2\int_0^t\int_{D_k} u(y,\tau)\langle\nabla h(y), \nabla_y q(t-\tau,y,x)\rangle\,d\mu(y)y\,d\tau \\
&\qquad - \int_0^t\int_{D_k} u(y,\tau)\Delta h(y)q(t-\tau,y,x)\,d\mu(y)\,d\tau.
\end{aligned}$$

Note that according to the definition of $h(y)$ the integration on the right hand side extends only over the annullar region where $D(x)-(2/3)\rho \leq d(x,y) \leq D(x)-(1/3)\rho$. In particular, $d(x,y) \geq D(x)/3 \geq \rho/3$ for such y. Donnelly proves ((see [Don1]) that the heat kernel q is almost euclidean, i.e. that one has in particular for all $t \in (0,T]$

$$|\nabla_y^l \nabla_y^m q(t,x,y)| \leq C_5 t^{-(n+m+l)/2} e^{-\frac{C_6 d^2(x,y)}{t}} \qquad \text{for } k,l = 0,1 \tag{2.2}$$

with constants C_5, C_6 depending only on the geometry of M and T. We use this together with the maximum principle applied to u and the monotonicity of $e^{-C_6 D^2(x)/s}$ as a function of s to obtain

$$u(x,t) \leq C_7 N e^{-C_8 D^2(x)/t}\mathrm{vol}\,(B_{D(x)}(x)).$$

Since the volume growth of $\widehat{M}$ is at most exponential, the lemma follows. □

To obtain Theorem 2.1 from the lemma we need an estimate of the form

$$\sup_{(x,y,t)\in\partial D_k\times(D_k\setminus B(\partial D_k)\times[0,T]} |k_j(t,x,y) - p_j^k(t,x,y)| \leq A, \tag{2.3}$$

with A independent of k. We have such a bound for $k_j(t,x,y)$ in (2.2) so it suffices to estimate p_j^k. We do this using the finite propagation speed method as in [CGT, Sections 1,2]. As observed by Chernoff [Chern], solutions ω of the wave equation on j-forms

$$\Delta_j\omega + \frac{\partial^2\omega}{\partial t^2} = 0$$

satisfying the initial condition $\frac{\partial\omega}{\partial t}(x,0) \equiv 0$ and appropriate boundary conditions have finite propagation speed i.e. if $\omega(x,0)$ is supported in $B_r(x)$, then $\mathrm{supp}\,\omega(x,t) \subset B_{r+t}(x)$. Chernoff considers only complete manifolds with boundary but his proof carries over verbatim in our context. As explained in [CGT], under appropriate bounded geometry assumptions, standard elliptic estimates (Sobolev and Gårding inequalities) and finite propagation speed give rise to pointwise estimates of the heat kernel. We review this briefly in our situation. Consider $\Delta = \Delta_j$ as an unbounded self-adjoint operator on L^2 j-forms on D_k corresponding to either relative or absolute boundary conditions. By spectral

theorem applied to $\sqrt{\Delta}$ and the Fourier inversion formula

$$\Delta^m e^{-t\Delta}\Delta^l u = \Delta^{m+l}e^{-t\Delta}u = \frac{1}{\pi}\int_0^\infty \frac{1}{\sqrt{2\pi t}}\frac{\partial^{2m+2l}}{\partial s^{2m+2l}}\left(e^{-\frac{s^2}{4t}}\right)\cos(s\sqrt{\Delta})u\,ds \tag{2.4}$$

for a fixed $t>0$. Of course we require here that u is in the domain of $\sqrt{\Delta}$, i.e. it should satisfy appropriate boundary conditions. Now $\cos(s\sqrt{\Delta})u$ satisfies the wave equation and the initial condition above so that, if $x_0 \in \partial D_k$, $y_0 \in D_k \setminus B(\partial D_k)$ and $\operatorname{supp} u \subset B_{\rho/3}(y_0)$, $\operatorname{supp}\cos(t\sqrt{\Delta})u \subset B_{\frac{\rho}{3}+t}(y_0)$. The formula (2.4) implies that

$$\begin{aligned}\| \ \Delta^m e^{-t\Delta}\Delta^l u \ \|_{L^2(B_{\rho/3}(x_0)} &\leq C_9 t^{-(\frac{1}{2}+2m+2l)} \ \| \ u \ \| \int_{\rho/3}^\infty e^{-\frac{s^2}{4t}}\,ds \\ &\leq C_{10}t^{-(\frac{1}{2}+2m+2l)}e^{-\frac{\rho^2}{72t}} \leq C_{11} \ \| \ u \ \|\end{aligned}$$

The kernel of $\Delta^m e^{-t\Delta}\Delta^l$ is $\Delta_x^m\Delta_y^l p_j^k(t,x,y)$, so that if

$$v(x,t) = \int_{D_k}\Delta^m p_j^k(t,x,y)u(y)\,d\mu(y),$$

the inequality above gives an L^2 bound for $\Delta^m v$. Summing over $m = 0,1\dots[N/4]+1$ an applying standard elliptic estimates (which hold with uniform constants due to our bounded geometry assumptions) we obtain an estimate for $|v(x_0,t)|$ i.e.

$$|\int_{D_k}\Delta^l p_j^k(t,x_0,y)u(y)d\mu(y)| \leq C_{11} \ \| \ u \ \|$$

valid for smooth u satisfying the boundary conditions and supported in $B_{\rho/3}(y_0)$. It follows that

$$\| \ \Delta^l p_j^k(t,x_0,y) \ \|_{L^2(B_{\rho/3}(y_0))} \leq C_{11}.$$

Now, applying interior elliptic estimates, we see again that

$$|p_j^k(t,x_0,y_0)| \leq C_{12}.$$

Since $x_0 \in \partial D_k, y_0 \in D_k \setminus B(\partial D_k)$ are arbitrary, (2.3) is proven.

Proof. (**Theorem 2.1**) By (2.1), $u(x,t) = e^{-\alpha t}|p_j^k(t,x,y) - k_j(t,x,y)|^2$ with y fixed satisfies the conditions of Lemma 2.2. Therefore

$$|p_j^k(t,x,y) - k_j(t,x,y)| \leq C_1 e^{-C_2\frac{D^2(y)}{t}}$$

by (2.3). The theorem follows from the symmetry of the heat kernels in x and y. □

In addition to the estimate above, we shall need a cheap bound of $|p_j^k(t,y,y)|$ for y near the boundary of D_k. We actually state one valid for all $y \in D_k$.

Lemma 2.3. *There exists a constant $c(t_0)$ which depends on the local geometry of M but is independent of k such that for all $y \in D_k$ and $t \geq t_0$*

$$|p_j^k(t,y,y)| \leq c(t_0).$$

Proof. Note that $x^{m+l}e^{-tx} \le c(m,l)t^{-m-l}$ for all $x, t > 0$. Thus, by the spectral theorem,

$$\| \Delta^m e^{-t\Delta}\Delta^l \| \le c(m,l)t^{-m-l}.$$

This can be converted into a pointwise bound for the kernel exactly as above and yields the lemma. □

3. Proofs of the main theorems

In this section, we shall also assume that $\Gamma \to \widehat{M} \to M$ is an noncompact Galois cover of a compact manifold M with an amenable covering group Γ and that $\{D_k\}_{k=1}^{\infty}$ is a regular exhaustion of $\widehat{M}$.

We begin with an approximation theorem for the heat kernels. A special case of this proposition settles a conjecture of Adachi and Sunada in [AdSu], section 1.

Proposition 3.1. *For every $t > 0$, we have*

$$Tr_\Gamma(e^{-t\Delta_j}) = \lim_{k\to\infty} \frac{vol(M)}{vol(D_k)} Tr(e^{-t\Delta_j^{(k)}}),$$

where Δ_j denotes the Laplacian acting on L^2 forms on $\widehat{M}$ and $\Delta_j^{(k)}$ denotes the Laplacian on D_k associated to either the relative boundary conditions or the absolute boundary conditions. The convergence is uniform in $t \in [t_0, t_1]$ for every $t_1 > t_0 > 0$.

Proof. First observe that

$$\mathrm{Tr}_\Gamma(e^{-t\Delta_j}) = \int_F \mathrm{tr}(k_j(t,x,x))d\mu(x) \tag{3.1}$$

$$= \lim_{k\to\infty} \frac{\mathrm{vol}(M)}{\mathrm{vol}(D_k)} \int_{D_k} \mathrm{tr}(k_j(t,x,x))d\mu(x) \tag{3.2}$$

where F is a nice (say with piecewise smooth boundary) fundamental domain for the action of Γ. To see this we denote by D_k' the union of those translates of fundamental domains which are contained in D_k and write

$$\mathrm{Tr}_\Gamma(e^{-t\Delta_j}) = \frac{\mathrm{vol}(M)}{\mathrm{vol}(D_k')} \int_{D_k'} \mathrm{tr}(k_j(t,x,x))d\mu(x) =$$
$$\frac{\mathrm{vol}(M)}{\mathrm{vol}(D_k)} \int_{D_k} \mathrm{tr}(k_j(t,x,x))d\mu(x) + \frac{\mathrm{vol}(M)}{\mathrm{vol}(D_k)} \int_{D_k\setminus D_k'} \mathrm{tr}(k_j(t,x,x))d\mu(x)$$
$$- \frac{\mathrm{vol}(M)\mathrm{vol}(D_k\setminus D_k')}{\mathrm{vol}(D_k)\mathrm{vol}(D_k')} \int_{D_k'} \mathrm{tr}(k_j(t,x,x))d\mu(x).$$

Since $D_k \setminus D_k' \subset D_{\mathrm{diam}(F)}$, the last two summands tend to zero as $k \to \infty$ by (0.1) and (2.2). This proves (3.1).

We now estimate using Theorem 2.1 and Lemma 2.3,

$$\begin{aligned}
& \frac{\mathrm{vol}(M)}{\mathrm{vol}(D_k)}\int_{D_k}\mathrm{tr}(k_j(t,x,x)-p_j^k(t,x,x))d\mu(x)\\
= \ & \frac{\mathrm{vol}(M)}{\mathrm{vol}(D_k)}\int_{\partial_\delta D_k}\mathrm{tr}(k_j(t,x,x)-p_j^k(t,x,x))d\mu(x)\\
+ \ & \frac{\mathrm{vol}(M)}{\mathrm{vol}(D_k)}\int_{D_k\backslash\partial_\delta D_k}\mathrm{tr}(k_j(t,x,x)-p_j^k(t,x,x))d\mu(x)\\
\leq \ & c(t_0)\frac{\mathrm{vol}(M)}{\mathrm{vol}(D_k)}\mathrm{vol}(\partial_\delta D_k)+\frac{\mathrm{vol}(M)}{\mathrm{vol}(D_k)}C_{11}e^{-\frac{C_{12}\delta^2}{t}}\mathrm{vol}(D_k).
\end{aligned}$$

Taking the limit as $k\to\infty$, we have

$$\limsup_{k\to\infty}\left|\mathrm{Tr}_\Gamma(e^{-t\Delta_j})-\frac{\mathrm{vol}(M)}{\mathrm{vol}(D_k)}\mathrm{Tr}(e^{-t\Delta_j^{(k)}})\right|\leq\mathrm{vol}(M)C_{11}e^{-\frac{C_{12}\delta^2}{t}}.$$

for arbitrary $\delta>0$, proving the proposition. $\square$

Remarks. A key result in the paper of Adachi and Sunada [AdSu] is a special case of the approximation property in Proposition 3.1 above, with Dirichlet boundary conditions and for functions (i.e. for the special case when $j=0$). They also conjecture in section 1 of their paper that such an approximation property should also hold for the spectral density functions, with Neumann boundary conditions. Proposition 3.1 above proves their conjecture, and much more, as complete results are obtained for differential forms, and not just for functions.

Proof. (of Theorem 0.1) We will only prove (b), as (a) is obtained from (b) by adding the 2 inequalities for $N=i$ and $N=i-1$. Recall that $\Delta_j^{(k)}$ has a discrete point spectrum for all $j\geq 0$, since the domain D_k is compact. Let λ be a positive eigenvalue of $\Delta^{(k)}$, and E_λ be the space of eigenforms with eigenvalue λ. Then E_λ decomposes into a direct sum $E_\lambda=E_\lambda^0\oplus\cdots\oplus E_\lambda^n$, where E_λ^j denotes the space of eigen j-forms of $\Delta_j^{(k)}$ with eigenvalue λ. Since $\lambda\neq 0$, the complex

$$0\to E_\lambda^0\xrightarrow{d}\cdots\xrightarrow{d}E_\lambda^n\to 0$$

is an *exact* sequence, so $D^N(\lambda)=\sum_{j=0}^N(-1)^{N-j}\dim E_\lambda^j$ is non-negative for any N such that $n\geq N\geq 0$. Also observe that

$$\sum_{j=0}^N(-1)^{N-j}\mathrm{Tr}(e^{-t\Delta_j^{(k)}})=\sum_{j=0}^N(-1)^{N-j}b^j(D_k,\partial D_k)+\sum_{\lambda\neq 0}e^{-t\lambda}D^N(\lambda).$$

Therefore we see that

$$\sum_{j=0}^N(-1)^{N-j}\mathrm{Tr}(e^{-t\Delta_j^{(k)}})\geq\sum_{j=0}^N(-1)^{N-j}b^j(D_k,\partial D_k).$$

Multiplying both sides of the previous inequality by $\frac{\mathrm{vol}(M)}{\mathrm{vol}(D_k)}$ and taking the limit superior as $k \to \infty$ and using Proposition 3.1, one sees that

$$\sum_{j=0}^{N}(-1)^{N-j}\mathrm{Tr}_\Gamma(e^{-t\Delta_j}) \geq \limsup_{k\to\infty}\sum_{j=0}^{N}(-1)^{N-j}\frac{\mathrm{vol}(M)}{\mathrm{vol}(D_k)}b^j(D_k,\partial D_k).$$

The proof of one of the inequalities part (b) is completed by taking the limit as $t \to \infty$. The other inequality in part (b) is similarly proved. □

Proof. (of Theorem 0.2)

We first prove the following "index" theorems, (see [J] for similar results)

$$\lim_{k\to\infty}\frac{\mathrm{vol}(M)}{\mathrm{vol}(D_k)}\chi(D_k,\partial D_k) = \chi(M) \tag{3.3}$$

and

$$\lim_{k\to\infty}\frac{\mathrm{vol}(M)}{\mathrm{vol}(D_k)}\chi(D_k) = \chi(M) \tag{3.4}$$

These are proved as follows. Let B denote either the absolute boundary conditions or the relative boundary conditions, and $\chi(D_k)_B$ denote either $\chi(D_k)$ (for the absolute boundary conditions) and $\chi(D_k,\partial D_k)$ (for the relative boundary conditions). Then one has the McKean-Singer identity (cf. [G] section 4.2), for$t > 0$,

$$\chi(D_k)_B = \sum_{j=0}^{n}(-1)^j\mathrm{Tr}(e^{-t\Delta_j^{(k)}}). \tag{3.5}$$

By Proposition 3.1, one sees that for $t > 0$,

$$\begin{aligned}
\lim_{k\to\infty}\frac{\mathrm{vol}(M)}{\mathrm{vol}(D_k)}\chi(D_k)_B &= \sum_{j=0}^{n}(-1)^j\lim_{k\to\infty}\frac{\mathrm{vol}(M)}{\mathrm{vol}(D_k)}\mathrm{Tr}(e^{-t\Delta_j^{(k)}}) \\
&= \sum_{j=0}^{n}(-1)^j\mathrm{Tr}_\Gamma(e^{-t\Delta_j}) \\
&= \chi(M)
\end{aligned}$$

where the last equality is the L^2 analogue of the McKean-Singer identity (cf. [Roe] chapter 13). This establishes the equalities (3.3) and (3.4). Note that when $n = \dim M$ is even, then $\chi(D_k) = \chi(D_k,\partial D_k)$ and when n is odd, then $\chi(D_k) = -\chi(D_k,\partial D_k) = \frac{1}{2}\chi(\partial D_k)$.

Let $n = 2$. Since Γ is infinite

$$b^0_{(2)}(\widehat{M},\Gamma) = 0.$$

By L^2 Poincaré duality (cf. [At], (6.4))

$$b^2_{(2)}(\widehat{M},\Gamma) = 0.$$

So that by Theorem 0.1, we see that Theorem 0.2 is true for $j \neq 1$. By (3.3) and (3.4), we see that

$$\begin{aligned}\lim_{k\to\infty} \frac{\mathrm{vol}(M)}{\mathrm{vol}(D_k)} b^1(D_k, \partial D_k) &= \chi(M) = \lim_{k\to\infty} \frac{\mathrm{vol}(M)}{\mathrm{vol}(D_k)} b^1(D_k) \\ &= b^1_{(2)}(\widehat{M}, \Gamma)\end{aligned}$$

where we have used the Euler characteristic property of L^2 Betti numbers (cf. section 1). This proves Theorem 0.2, part a).

Let $n = 3, 4$. By a result of Cheeger and Gromov (cf. [CG1] Lemma 3.1), the natural forgetful map

$$H^1_{(2)}(\widehat{M}) \hookrightarrow H^1(\widehat{M})$$

is injective. Since Γ is infinite, we know that $H^1_{(2)}(\widehat{M})$ is infinite dimensional if $b^1_{(2)}(\widehat{M}, \Gamma) \neq 0$. By hypothesis $\dim H^1(\widehat{M}) < \infty$, therefore

$$b^1_{(2)}(\widehat{M}, \Gamma) = 0.$$

Since Γ is infinite

$$b^0_{(2)}(\widehat{M}, \Gamma) = 0.$$

Now let $n = 3$. By L^2 Poincaré duality (cf. [At], (6.4))

$$b^3_{(2)}(\widehat{M}, \Gamma) = 0 = b^2_{(2)}(\widehat{M}, \Gamma).$$

So that by Theorem 0.1, we see that Theorem 0.2 is true when $n = 3$.

Now let $n = 4$. By L^2 Poincaré duality (cf. [At], (6.4))

$$b^4_{(2)}(\widehat{M}, \Gamma) = 0 = b^3_{(2)}(\widehat{M}, \Gamma).$$

So that by Theorem 0.1, we see that Theorem 0.2 is true for $j \neq 2$. By (3.3) and (3.4), we see that

$$\begin{aligned}\lim_{k\to\infty} \frac{\mathrm{vol}(M)}{\mathrm{vol}(D_k)} b^2(D_k, \partial D_k) &= \chi(M) = \lim_{k\to\infty} \frac{\mathrm{vol}(M)}{\mathrm{vol}(D_k)} b^2(D_k) \\ &= b^2_{(2)}(\widehat{M}, \Gamma)\end{aligned}$$

where we have used the Euler characteristic property of L^2 Betti numbers (cf. section 1). This proves Theorem 0.2.

$\square$

We now give an example to show that the analogue of Theorem 0.2 is false (and hence so is the Conjecture) for certain non-amenable covering spaces.

Example. Consider the $2n$-dimensional hyperbolic space $\mathbb{H}^{2n}$ and the sequence of open balls $\{D_k\}_{k=1}^\infty$, where $D_k = B_k(x_0)$ denotes the ball of radius k which is cetered at $x_0 \in \mathbb{H}^{2n}$. Then, by an explicit calculation, $\{D_k\}_{k=1}^\infty$ is a regular exhaustion, i.e.

$$\lim_{k\to\infty} \frac{\mathrm{vol}_{n-1}(\partial D_k)}{\mathrm{vol}_n(D_k)} = h(\mathbb{H}^{2n}) = \frac{(2n-1)^2}{4} \neq 0.$$

Let Γ be a uniform lattice in $\mathbb{H}^{2n}$. It follows that $\Gamma \to \mathbb{H}^{2n} \to M$ is a *non-amenable* Galois covering space, where $M = \mathbb{H}^{2n}/\Gamma$. It is a classical calculation that (see [Don2]) that

$$\mathcal{H}^j(\mathbb{H}^{2n}) = \begin{cases} \{0\} \text{ if } j \neq n; \\ \\ \text{infinite dimensional if } j = n. \end{cases}$$

where $\mathcal{H}^j(\mathbb{H}^{2n})$ denotes the space of L^2 harmonic j-forms on $\mathbb{H}^{2n}$. It follows that

$$b^n_{(2)}(\mathbb{H}^{2n}, \Gamma) \neq 0.$$

However, since D_k is topologically a ball in $\mathbb{R}^{2n}$, one sees that for all $k \geq 1$,

$$H^n(D_k) = \{0\} = H^n(D_k, \partial D_k).$$

Therefore

$$\lim_{k\to\infty} \frac{\mathrm{vol}(M)}{\mathrm{vol}(D_k)} b^n(D_k) = 0 < b^n_{(2)}(\mathbb{H}^{2n}, \Gamma)$$

and

$$\lim_{k\to\infty} \frac{\mathrm{vol}(M)}{\mathrm{vol}(D_k)} b^n(D_k, \partial D_k) = 0 < b^n_{(2)}(\mathbb{H}^{2n}, \Gamma).$$

□

Proof. (of Theorem 0.3) We first prove part a). The proof uses the hypothesis to justify the interchange of the limits as $k \to \infty$ and as $t \to \infty$. We write

$$\Big| \frac{\mathrm{vol}(M)}{\mathrm{vol}(D_k)} b^j(D_k, \partial D_k) - b^j_{(2)}(\widehat{M}, \Gamma) \Big|$$

$$\leq f_j(t) + \Big| \mathrm{Tr}_\Gamma(e^{-t\Delta_j}) - \frac{\mathrm{vol}(M)}{\mathrm{vol}(D_k)} \mathrm{Tr}(e^{-t\Delta_j^{(k)}}) \Big| + \Big| \mathrm{Tr}_\Gamma(e^{-t\Delta_j}) - b^j_{(2)}(\widehat{M}, \Gamma) \Big|.$$

Taking the limit as $k \to \infty$ and using Corollary 3.1, one has

$$\Big| \lim_{k\to\infty} \frac{\mathrm{vol}(M)}{\mathrm{vol}(D_k)} b^j(D_k, \partial D_k) - b^j_{(2)}(\widehat{M}, \Gamma) \Big| \leq f_j(t) + \Big| \mathrm{Tr}_\Gamma(e^{-t\Delta_j}) - b^j_{(2)}(\widehat{M}, \Gamma) \Big|.$$

Since this is true for all $t \geq 1$, the Theorem follows by taking the limit as $t \to \infty$ and using the hypothesis that $\lim_{t\to\infty} f_j(t) = 0$. The proof of part b) is similar.

□

4. Miscellaneous results on the spectrum of the Laplacian

In this section, we derive some corollaries to the main theorems, as well as several results on the spectrum of the Laplacian and zeta functions associated to the Laplacian. We shall continue to assume that $\Gamma \to \widehat{M} \to M$ is an noncompact Galois cover of a compact manifold M with an amenable covering group Γ and that $\{D_k\}_{k=1}^\infty$ is a regular exhaustion of $\widehat{M}$.

Let $N_j^{(k)}(\lambda)$ denote the spectral function for $\Delta_j^{(k)}$, i.e. $N_j^{(k)}(\lambda)$ is the number of eigenvalues of $\Delta_j^{(k)}$ which are less than or equal to λ. Let $N_{j,\Gamma}(\lambda)$ denote the von Neumann's spectral function for Δ_j (see section 1). Let $\sigma(\Delta_j)$ denote the spectrum of Δ_j and $\sigma(\Delta_j^{(k)})$ denote the spectrum of $\Delta_j^{(k)}$. Then one has the following corollary of Proposition 3.1.

Corollary 4.1. *Let $\widehat{M}$ be an amenable Galois covering.*

a) *At all points λ which are points of continuity of $N_{j,\Gamma}(\lambda)$, one has*

$$\lim_{k\to\infty} \frac{vol(M)}{vol(D_k)} N_j^{(k)}(\lambda) = N_{j,\Gamma}(\lambda).$$

b) $\sigma(\Delta_j) \subset \overline{\bigcup_{k\geq 1} \sigma(\Delta_j^{(k)})}$

Proof. Part a) is immediately deduced from Proposition 3.1, also using a Lemma due to Shubin (cf. [Sh]). Now let λ_1 and λ_2 be points of continuity of $N_{j,\Gamma}$, and $\lambda_1 < \lambda_2$. Then by part a), one has

$$\lim_{k\to\infty} \frac{\mathrm{vol}(M)}{\mathrm{vol}(D_k)} \left[N_j^{(k)}(\lambda_2) - N_j^{(k)}(\lambda_1) \right] = N_{j,\Gamma}(\lambda_2) - N_{j,\Gamma}(\lambda_1).$$

From this, one easily deduces part b). □

Remark 4.2. The limit $\lim_{k\to\infty} \frac{\mathrm{vol}(M)}{\mathrm{vol}(D_k)} N_j^{(k)}(\lambda)$ is refered to in the Mathematical Physics literature as the *integrated density of states* for Δ_j. The corollary above can be viewed as stating that the integrated density of states for Δ_j is independent of the choice of boundary conditions (relative or absolute).

The closed subspaces $\overline{d\Omega_c^{p-1}(\widehat{M})}$ and $\overline{\delta\Omega_c^{p+1}(\widehat{M})}$ of $\Omega^p_{(2)}(\widehat{M})$ are preseved by the Laplacian Δ_p. Let $G_p(\lambda)$ and $F_p(\lambda)$ denote the respective spectral density functions, i.e.

$$G_p(\lambda) = \mathrm{Tr}_\Gamma(\chi_{[0,\lambda]}(\Delta_p^{<1>}))$$

and

$$F_p(\lambda) = \mathrm{Tr}_\Gamma(\chi_{[0,\lambda]}(\Delta_p^{<2>}))$$

where $\Delta_p^{<1>}$ denotes the retriction of Δ_p to the closed subspace $\overline{d\Omega_c^{p-1}(\widehat{M})}$ and $\Delta_p^{<2>}$ denotes the retriction of Δ_p on the closed subspace $\overline{\delta\Omega_c^{p+1}(\widehat{M})}$ and $\chi_{[0,\lambda]}$ denotes the characteristic function of the closed interval $[0,1]$. Then the following can be viewed as a generalization of one of the results in [CG1].

Proposition 4.3. *Let $\widehat{M}$ be an amenable Galois covering and $\sigma(\Delta_p)$ denote the spectrum of the Laplacian acting on L^2 p-forms on $\widehat{M}$. Then $0 \in \sigma(\Delta_1)$.*

Proof. We observe the following obvious characterization: $0 \in \sigma(\Delta_p)$ if and only if $N_p(\lambda) > 0$ for all $\lambda > 0$. Since Γ is amenable, one easily sees that $0 \in \sigma(\Delta_0)$, therefore $N_0(\lambda) > 0$ for all $\lambda > 0$. By Lemma 3.1 in [GS], one sees that $F_p(\lambda) = G_{p+1}(\lambda)$ for all $p \geq 0$. So $N_1(\lambda) \geq G_1(\lambda) = F_0(\lambda) = N_0(\lambda) > 0$, i.e. $0 \in \sigma(\Delta_1)$. □

The following is a corollary of Theorem 0.3, noting that the large time behaviour of the heat kernel corresponds to the small λ behaviour of the corresponding spectral density function.

Corollary 4.4. *Let $\widehat{M}$ be an amenable Galois covering. Suppose that there is a positive constant C such that for all $k \in \mathbb{N}$, one has*

$$\frac{vol(M)}{vol(D_k)}\Big[N_j^{(k)}(\lambda) - N_j^{(k)}(0)\Big] \leq C\lambda^{\beta_j}$$

for all $\lambda \in (0,1)$. Then the j^{th} Novikov-Shubin invariant (cf. [GS]*) $\alpha_j(\widehat{M}) \geq \beta_j$ and the conjecture in the introduction is true.*

We now discuss zeta functions and convergence questions.

Let $\lambda > 0$ and $\zeta(s, \Delta_p^{(k)} + \lambda)$ denote the zeta function of the operator $\Delta_p^{(k)} + \lambda$, upto normalization, that is,

$$\zeta(s, \Delta_p^{(k)} + \lambda) = \frac{\mathrm{vol}(M)}{\mathrm{vol}(D_k)} \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}\big(\mathrm{Tr}(e^{-t(\Delta_p^{(k)}+\lambda)}) - e^{-t\lambda} b^p(D_k, \partial D_k)\big)dt$$

in the case of relative boundary conditions, and in the case of absolute boundary conditions, one replaces $b^j(D_k, \partial D_k)$ by $b^j(D_k)$. Then standard arguments [RS] show that $\zeta(s, \Delta_p^{(k)} + \lambda)$ is a holomorphic function of s in the half-plane $\Re(s) > n/2$, and it has a meromorphic extension to $\mathbb{C}$ with no pole at $s = 0$.

Also let $\lambda > 0$ and $\zeta(s, \Delta_p + \lambda)$ denote the L^2 zeta function of the operator $\Delta_p + \lambda$, that is,

$$\zeta(s, \Delta_p + \lambda) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}\big(\mathrm{Tr}_\Gamma(e^{-t(\Delta_p+\lambda)}) - e^{-t\lambda} b^p_{(2)}(\widehat{M})\big)dt.$$

Then arguments as in [M] show that $\zeta(s, \Delta_p + \lambda)$ is a holomorphic function of s in the half-plane $\Re(s) > n/2$, and it has a meromorphic extension to $\mathbb{C}$ with no pole at $s = 0$. Then the following proposition is relatively straightforward consequence of Proposition 3.1.

Proposition 4.5. *For $\lambda > 0$ fixed and as $k \to \infty$, $\zeta(s, \Delta_p^{(k)} + \lambda)$ converges to $\zeta(s, \Delta_p + \lambda)$, where the convergence is uniform on compact subsets of $\Re(s) > n/2$.*

If the following question can be answered in the affirmative, then one can also deduce convergence of determinants as $k \to \infty$.

Question. For $\lambda > 0$ fixed and as $k \to \infty$, does $\zeta(s, \Delta_p^{(k)} + \lambda)$ converge to $\zeta(s, \Delta_p + \lambda)$, uniformly on compact subsets near $s = 0$?

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